4. Termination of TRSs

Termination of TRSs needed for
- word problem
- confluence checking (for terminating TRSs, confluence is decidable)
- program verification
- induction proofs

Contents:

4.1. Noetherian Induction

named after Emmy Noether

Up to now: Induction on numbers, terms, positions, etc.

These are all special cases of a more general induction principle which relies on the connection between induction and termination.

- For natural numbers:

  To prove $\forall x \in \mathbb{N}.\ \varphi(x)$

  it suffices to prove

  $$\varphi(0) \qquad \text{(Ind. Base)}$$

  $$\forall y \in \mathbb{N}.\ \underbrace{\varphi(y)}_{\text{Ind. Hypothesis}} \Rightarrow \varphi(y+1) \qquad \text{(Ind. Step)}$$

- Similar principles for other data structures

<u>Goal</u>: Generalize this induction principle

- arbitrary sets M (instead of $\mathbb{N}$, $\mathcal{T}(\Sigma, \mathcal{V})$, $\mathbb{N}^2$, ...)
- arbitrary well-founded induction relations $\succ$

Idea: When proving φ for some object $m \in M$, we can assume as induction hypothesis that φ already holds for all $k \in M$ that are smaller than m (i.e., where $m \succ k$).

To prove $\quad \forall n \in M.\ \varphi(n)$

it suffices to show

$$\forall m \in M. \ \underbrace{\left(\forall k \in M.\ m \succ k \Rightarrow \varphi(k)\right)}_{\text{Ind. Hypothesis}} \Rightarrow \varphi(m)$$

The other induction principles are special cases of Noetherian induction:

- $M = \mathbb{N}$ where $m \succ k$ iff $m = k+1$

  Ind. Base $\widehat{=}$ elements that have no smaller elements w.r.t. the induction relation $\succ$

- $M = \mathcal{T}(\Sigma, \mathcal{V})$ where $m \succ k$ iff $m = f(t_1, \dots, t_n)$ and k is a direct subterm of m (i.e., $k \in \{t_1, \dots, t_m\}$).

In addition, there are many more well-founded relations on $\mathbb{N}$, $\mathcal{T}(\Sigma, \mathcal{V}), \dots \Rightarrow$ we obtain many possible induction principles.

<u>Def 4.1.1.</u> (Noetherian Induction)

Let $\succ$ be a well-founded relation on a set M.

For all $m \in M$, let the following hold:

if $\varphi(k)$ holds for all $k \in M$ with $m \succ k$, then $\varphi(m)$ holds as well.

Then $\varphi(n)$ holds for all $n \in M$.

<u>Thm 4.1.2</u> (Correctness of Noetherian Induction)

Noetherian induction is correct.

<u>Proof</u>: Assume that

$\forall m \in M. \left(\forall k \in M.\ m \succ k \Rightarrow \varphi(k)\right) \Rightarrow \varphi(m)$ holds,

but $\forall n \in M.\ \varphi(n)$ does not hold.

So there is a counterexample $n_0 \in M$ with $\neg\varphi(n_0)$.

But: $\left(\forall k \in M\ n_0 \succ k \Rightarrow \varphi(k)\right) \Rightarrow \varphi(n_0)$

Therefore, there must be a smaller counterexample than n_0, i.e., there is an n_1 with $n_0 \succ n_1$ with $\neg\varphi(n_1)$.

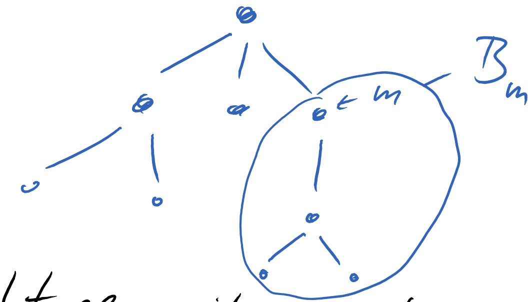
Analogously, there must be a smaller counterexample than n_1, i.e., there is an n_2 with $n_0 \succ n_1 \succ n_2$ with $\neg\varphi(n_2)$.

In this way, we generate an infinite decreasing sequence of counterexamples $n_0 \succ n_1 \succ n_2 \succ \ldots$ which contradicts well-foundedness of $\succ$. ☒

This needs the "axiom of choice" that states choosing infinitely many times is possible.

The following lemma is an example for an application of Noetherian induction (and the lemma is needed in Sect 4.2).

<u>Thm 4.1.3</u> (Lemma of König)
A tree with finite branching factor (i.e., every node has finitely many children) where each path is finite only has finitely many nodes.



<u>Proof</u>
Let M be the set of all nodes.
For every $m \in M$, let B_m be the subtree with root m.
For $m, k \in M$ let $m \succ k$ iff k is a direct child of m.
The relation $\succ$ is well founded, because all paths are finite.
We want to prove the following statement $\varphi(n)$ for all nodes $n \in M$:

<u>B_n only has finitely many nodes.</u>

This is sufficient for the Thm, because then φ also holds for the root node of the tree.

Let $m \in M$.

$$|B| = 1 + \sum |B_k| \qquad (*)$$

number of nodes in B_m

k is child of m

By Noetherian induction, the ind. hypothesis states that $|B_k|$ is finite for all children $m > k$.
As m only has finitely many children, (*) implies that $|B_m|$ is also finite. $\boxed{\text{E}}$

Well-foundedness $\cong$ Termination

$\leadsto$ Strong Connection between Termination and Induction.